RESEARCH STATEMENT

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1. Overview

My research interest lies in number theory as well as in representation theory. In particular, much of my research is related to the Langlands program.

The Langlands program is a web of conjectures on connections between arithmetics and representation theory. For example, the global Langlands correspondence, which is regarded as a central conjecture, aims to build a connection between automorphic representations of $G(F)$ (representation theory) and the representations a conjectured Langlands group L_F (arithmetics). Note that the term representation is slightly abused to be a homorphism from the Langlands group L_F to the L-group ${}^LG$ of G. Under the philosophy of connecting the two distant theories, the research field is rich in topics.

1.1. **Explicit construction** [BX24]**.** One of my works involves explicating the local Langlands correspondence under a specific setting. More specifically, I dealt with the case when the group is the even quasi-split unitary group over a p-adic field and the representation is simple supercuspidal, including the dyadic case ($p = 2$). The tool I rely on is the Rankin-Selberg integrals.

The local Langlands correspondence claims the existence of a surjective map

$$\mathrm{LLC}_G : \Pi(G) \longrightarrow \Phi(G),$$

between the equivalent classes $\Pi(G)$ of irreducible admissible representation and the equivalent classes of Langlands parameters, whose fibers are finite and who preserves the local factors. It is already shown in many cases, including the unitary group case which my work is on. However, further efforts are required to explicate the correspondence.

The simple supercuspidal representations of reductive groups are the class of supercuspidal representations of minimal positive depth. Such representations for arbitrary reductive groups were developed and studied in [GR10] and generalized to a class of supercuspidal representations called the epipelagic representations in [RY14]. Simple supercupsidal representations are important towards obtaining results on arbitrary supercuspidal representations.

The Langlands parameter of a simple supercuspidal representation of various classical groups have been studied via different methods recently. Among the methods, one is to compute the rank-1 local factor and to observe its polar information [Adr16, AL16, AK19, AK21, AHKO23]. In particular, this is the only known method to study the dyadic case.

In my work [BX24], I computed the local gamma factor for the even unitary group using the Rankin-Selberg method. The computation showed us a holomorphic local gamma factor, similar to the odd orthogonal case in [Adr16]. Hence it is expected that the Langlands parameter being irreducible and the endoscopic lift being simple supercuspidal again, as studied in [Oi19] for the non-dyadic case.

1.2. **Relative version** [WX24]**.** This work aims to understand the Sp_{2n}-distinguished spectrum of $L^2([\mathrm{U}_{2n}])$, by studying the period integral of the pseudo-Eisenstein series. The project lies in the theory of Sakellaridis-Vekatesh known as the relative version of the Langlands program.

In general, let (G, H) be a pair of a reductive group G and its reductive subgroup H over a non-Archimedean local field F, where $X = G/H$ is a spherical variety. They conjecture that the local H-distinction of a representation π of G implies a functorial lift to a representation of another group G'. Although the global conjecture is not even well stated, we still expect a similar story. In particular, we expect the absence of the cuspidal distinguished spectrum when (G, H) is a vanishing pair.

A cuspidal subrepresentation $\pi \subseteq \mathcal{A}_0(G)$ is called H-distinguished if it vanishes under the linear functional of taking period integral over $[H]_G$, which is always well defined [AGR93]. Unlike the cuspidal case, the period integrals of an automorphic form is not always convergent. There is not yet a commonly acknowledged definition of the H-distinguished spectrum in general. Lapid and Offen gave a definition of the H-distinguished spectrum in a distributional way and studied the pair $(\mathrm{Sp}_{4n}, \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n})$ over a number field in [LO18]. Using the formula they derived on the period integrals of pseudo-Eisenstein series, they provided an upper bound of the distinguished spectrum (i.e. a space containing the spectrum) and constructed a lower bound of the discrete distinguished spectrum. Following their idea, we studied the distinguished spectrum for the pair $(\mathrm{U}_{2n}, \mathrm{Sp}_{2n})$. We derived a formula on the period integral of pesudo-Eisenstein series and determined an upper bound of the distinguished spectrum using the residue data. We aim to give another upper bound using the discrete data and construct a lower bound.

It is worth mentioning this work dates back to the work of Jacquet, Lapid and Rogawski, on the pair $(\mathrm{Res}_{E/F}\,\mathrm{GL}_n, \mathrm{GL}_n)$ and Galois pairs [JLR99, LR03].

2. L-parameters of simple supercuspidal representations: even unitary group[BX24]

Let E/F be an unramified quadratic extension of nonarchimedean local fields of characteristic 0 and residue characteristic $p > 0$. Let l be a positive integer and let U_{2l} be the unitary group on $2l$ variables with respect to the field extension E/F. Fix a level one additive character ψ_F and let $\psi = \mathrm{tr}_{E/F} \circ \psi_F$. The addtive character ψ of E is also of level one. Let τ be an unramified quasi-character of $E^\times$, and let π be a simple supercuspidal representation. We computed the Rankin-Selberg gamma factor $\gamma(s, \Upsilon, \pi \times \tau, \psi)$, due to a well-chosen data for the Rankin-Selberg integrals. The computation shall have an application on determining the Langlands parameter of the representation.

2.1. **The simple supercuspidal representations.**

The simple supercuspidal representations of reductive groups are the class of supercuspidal representations of minimal positive depth. In particular, they are well parametrized by simple data. In the case of unitary group, such a representation is parametrized by a character ω^1 on k_E^1 and $b \in k_F^\times$ where k_F is the residue field of F and k_E^1 is the group of norm one elements in the residue field k_E of E.

2.2. **The Rankin-Selberg gamma factor.**

The Rankin-Selberg integral for the unitary group is an integral representation of the local L-factors. It was defined by Ben-Artzi and Soudry [BAS09, BAS16]. By showing the space of the Rankin-Selberg integrals is one dimensional, Morimoto defined the Rankin-Selberg gamma factors as the quotient of the integrals, and showed its coincidence with Shahidi's gamma factor, which admits functoratlity properties.

Given a pair of generic representaions π and τ of U_{2l} and $\mathrm{GL}_n(E)$ respectively, with respect to a choice of generic characters, the Rankin-Selberg gamma factor is defined as the quotient of two Rankin-Selberg integrals, up to a normalizing factor. In more details, the Rankin-Selberg integral $\mathcal{L}(W, f_s, \phi)$ is defined for the data of a Whittaker function W in the Whittaker model of π, a holomorphic section in the principal series induced by τ and a Schwartz function ϕ (in the context of the Weil representation of the Heisenberg group).

We then consider the intertwining operator $M(\tau, s)$ with respect to the unique nontrivial Weyl group element, and normalize it using the local coefficient as $M^*(\tau, s) = C(\tau, s)M(\tau, s)$ i.e. normalize it to be such that the following diagram

$$\begin{array}{ccc} V(\tau, s) & \xrightarrow{M^*(\tau,s)} & V(\tau^*, 1-s) \\ & {\scriptstyle \Lambda(\tau,s)} \searrow \quad \swarrow {\scriptstyle \Lambda(\tau^*,1-s)} & \\ & \mathbb{C} & \end{array}$$

is commutative, where $\Lambda(\tau, s)$ and $\Lambda(\tau^*, s)$ are the standard Whittaker funcionals.

The space of such integrals form a one dimensional vector space [Mor23]. Consequently, the Rankin-Selberg gamma factor is defined as follows:

$$\mathcal{L}^*(W, f_s, \phi) = \gamma_0(s, \pi \times \tau, \Upsilon, \psi)\mathcal{L}(W, f_s, \phi).$$

It is further normalized as

$$\gamma(s, \pi \times \tau, \Upsilon, \psi) = \omega_\pi(-1)\tau(-1)^l \gamma_0(s, \pi \times \tau, \Upsilon, \psi)$$

where ω_π is the central character of the representation π. The gamma factor coincides with Shahidi's gamma factor in the sense (cf. [Mor23])

$$\gamma(s, \pi \times \Upsilon, \psi) = \gamma^{\mathrm{Sh}}(s, (\pi \otimes \Upsilon) \times \tau, \psi).$$

2.3. **The computation.** The techical heart of this work lies in the computation of the Rankin-Selberg gamma factors. By a well-chosen data of a Whittaker function W, holomorphic section f_s and a Schwartz function ϕ, the Rankin-Selberg integrals $\mathcal{L}(W, f_s, \phi)$ and $\mathcal{L}(W, M^* f_s, \phi)$ are computable (See §3.2).

Together with the computation of the local coefficient $C(\tau, s)$, we have

$$\gamma(s, \pi \times \tau, \Upsilon, \psi) = \omega^1(-1)\tau(-1)^l \tau(b^{-1}\varpi) q_F^{-2s+1},$$

and by its coincidence with the Shahidi's gamma factor [KK05, Mok15], we have

$$\gamma^{\mathrm{Sh}}(s, \pi \times \tau, \psi) = -\omega^1(-1)\tau(-1)^l \tau(b^{-1}\varpi) q_F^{-2s+1}.$$

2.4. **The application.** The gamma factor is holomorphic, which is similar to the case of odd orthogonal group [Adr16]. A similar argument in *loc. cit.* shall determine the Langlands parameters for large residue characteristic p. Indeed, this is already obtained by Oi for all odd p [Oi19].

In general, the computation of the Rankin-Selberg integrals barely has any obligation in the dyadic case. Moreover, the recent work [AHKO23] provides with us a powerful method of applying the computation to the determination of the parameter to the $p = 2$ case. Based on the work, we expect Oi's result to hold for all p, that is, the endoscopic lifts are still simple supercuspidal (and the Langlands parameter are irreducible).

3. Sp_{2n}-Distinction U_{2n} spectrum [WX24]

Fix an $n \in \mathbb{N}$ and let $(G, H) = (\mathrm{U}_{2n}, \mathrm{Sp}_{2n})$. We use the notation P for a standard Levi subgroup of G and let $P = MU$ be its standard Levi decomposition.

The technical core of our work is the formula on the period integral of pseudo-Eisenstein series, as in [LO18]. By unfolding the period integral, we get a sum over the double cosets $P\backslash G/H$. With the orbit analysis and the recognition of some vanishing pairs, we see the period integral is vanishing if M is not contained in the Siegel Levi subgroup. When M is contained in the Siegel Levi subgroup, by reinterpreting the double coset, the sum is actually over the Weyl group elements represented by $N_G(M) \cap X$, where $X = G \cdot e$ is a symmetric space embedded in G as a subvariety and is isomorphic to G/H. We get an upper bound of the distinguished spectrum based on this formula.

3.1. **Orbit Representatives.** A standard Levi subgroup M is always of a diagonal embedding of the groups $\mathrm{Res}_{E/F} \mathrm{GL}_{n_i}$ and U_{2m}. Given a Levi subgroup M and a Weyl group element $w \in W(M, M)$ preserving the Levi subgroup M, we consider the smallest semi-standard Levi subgroup $L = L(w)$ containing M and w. We are technically interested in the description of L_x and M_x when M is contained in the Siegel Levi subgroup and $x \in N_G(M) \cap X$ is M-minimal.

In [MO21], Mitra and Offen gave an explicit description of M_x when x is M-minimal as

$$M_x \cong \prod_{i \in S} \mathrm{GL}_{n_i} \times \prod_{i \in R} \mathrm{Res}_{E/F} \mathrm{GL}_{n_i} \times \prod_{i=\ell+1}^{k} \mathrm{Sp}_{n_i} \times \mathrm{U}_{2m},$$

where the index sets R and S are given by the M-minimal x. Following their work, we get an explicit description of L and its stabilizer when M is contained in the Siegel Levi subgroup. In particular, we have

$$L_x \cong \prod_{i \in S} \mathrm{GL}_{n_i} \times \prod_{i \in R} \mathrm{GL}_{2n_i}.$$

It is worth mentioning that we also need to consider the orbits when x is not M-minimal in the later work regarding the period integral and the intertwining period. Instead of study the stablizers directly, we use a reductive idea based the combinatorics of a graph attached to the group U_{2n} in [O17], which is defined in the spirit of [LR03] for Galois pairs.

3.2. **The Period Integral.** The main result we get is a formula of the period integral of pseudo-Eisenstein series induced from a standard Levi subgroup M contained in the Siegel Levi subgroup. The reason to ignore other cases is that the corresponding period integrals vanish. In order to see this, we need to understand how the orbit analysis is involved.

Given the convergence of the integrals over each orbit (cf. [AGR93]), the period integral decomposes into a sum of integrals indexed by the double coset $P\backslash G/H$. The integral associated to each orbit admits a further expression in terms of intertwining periods. In order to study the integral over each orbit, we technically identify the double coset $P\backslash G/H$ as $P\backslash X$. By removing some vanishing orbits and reinterpreting the rest, the index set is the set of M-orbits $N_G(M) \cap X/M$. Recall the orbit analysis we introduced in the previous section, that the pair of (M, M_x) is a vanishing pair when M is not contained in the Siegel

Levi subgroup since the factor in the central block $(\mathrm{U}_{2m}, \mathrm{Sp}_{2m})$ is vanishing. As we can see the integral has an inner integral over the relative adelic quotient $[M_x]_M$ of a cuspidal function, the period integrals are vanishing when M is not contained in the Siegel Levi subgroup.

Assuming that M is contained in the Siegel Levi subgroup, the further part of the formula claims that each orbit integral can be expressed in terms of intertwining periods as

$$I_x(\phi) = \int_{\lambda_x + i(\mathfrak{a}_M^*)_x^-} J(\phi[\lambda], x, \lambda) d\lambda.$$

The derivation of this formula comes from classical results in Fourier analysis, but only with the convergence of the intertwining periods.

3.3. **The Intertwining Periods.** The convergence of an intertwining period $J(\varphi, x, \lambda)$ involves technical properties of the integrand and orbit analysis. This section focuses on how the orbit analysis contributes to the proof.

Let M be contained in the Siegel Levi subgroup and x be M-minimal. By transforming the orbit integral, we get a double integral where the outer one is over $M_x(\mathbb{A})\backslash L_x(\mathbb{A})$ and the inner one over the relative adelic quotient $[M_x]_M$, and the integrand is the product of some characters on $L_x(\mathbb{A})$ and $M_x(\mathbb{A})$ and a function θ_f^M which is an bound to the rapidly decreasing function φ.

According to the orbit analysis, the pair (M, M_x) is isomorphic to the product of pairs of forms (again, under the assumption that M is contained in the Siegel Levi subgroup):

- $(\mathrm{Res}_{E/F} \mathrm{GL}_{n_i}, \mathrm{GL}_{n_i})$;
- $(\mathrm{Res}_{E/F}(\mathrm{GL}_{n_i} \times \mathrm{GL}_{n_i}), \mathrm{Res}_{E/F} \mathrm{GL}_{n_i})$, where the latter group embeds into the first one as $g \mapsto \mathrm{diag}(g, {}^t g^{-1})$.

It is a classical result that pairs of both of the above types admit a relative adelic quotient of finite volume. Hence for the product (M, M_x). Meanwhile, the integrand of the inner integral is well bounded and the outer integral turns out to be bounded by a constant multiple of a character of $L_x(\mathbb{A})$.

According to the orbit analysis again, the only non-trivial factor of the pair (L_x, M_x) is of the type $(\mathrm{GL}_{2n}, \mathrm{Res}_{E/F} \mathrm{GL}_n)$. After applying a conjugation to the integral, the convergence of the intertwining period reduces to a lemma of Jacquet, Lapid and Rogawski (cf. Lemma 27, [JLR99]).

3.4. **The Distinguished Spectrum.** The H-distinguished spectrum in the cuspidal subspace $L_0^2([G])$ is defined to be the space of those with nonvanishing period integrals. While the convergence of a period integral of a cusp form is guaranteed in general due to [AGR93], there is no such theorem for a general automorphic form. As a consequence, we study the period integral of pseudo-Eisenstein series and get the spectral information in the distributional sense.

We use the definition of the the H-distinguished spectrum $L^2_{H-\mathrm{dist}}([G])$ in [LO18], as the orthogonal complement to the space of pesudo-Eisenstein series with a vanishing period integral. The formula, under the setting of the finer decomposition, immediately implies an upper bound with respect to the residue data:

$$L^2_{H-\mathrm{dist}}([G]) \subseteq \hat{\bigoplus_{(\mathfrak{X}, \mathfrak{C}) \in \mathfrak{B}_{H-\mathrm{dist}}}} L^2_{\mathfrak{X}}([G])_{\mathfrak{C}},$$

where $\mathfrak{B}_{H-\mathrm{dist}}$ is defined in terms of the vanishing subspaces of the intertwining periods.

4. Current Projects

4.1. **Explicit formula for the unramified Bessel function of the odd general spin group.** The study of explicit unramified formulas date back to the famous Casselman-Shalika formula, which is on the unramified (also referred to as *spherical*) Whittaker function for a quasi-split reductive groups [Ca80, CS80]. Not only the formula is provided in their paper, the method they use in the papers, now widely known as the Casselman-Shalika method, is power in similar problems. Given an unramified principal series representation over a general or specific quasi-split reductive group over a p-adic field, one can ask for the explicit formula for other models (Waldspurger model, Bessel model, Shalika model, etc.). In particular, the unramifed Waldspurger function of GL_2 and the unramified Bessel function of the split odd orthogonal group SO_{2n+1} were computed in [BFF97].

The split general spinor group GSpin_{2n+1} is defined in terms of a root datum. Although it lacks a good matrix group representation, it fits well an exact sequence:

$$1 \to \mathbb{G}_m \to \mathrm{Gspin}_{2n+1} \to \mathrm{SO}_{2n+1} \to 1,$$

in the sense the structures are well preserved under the projection.

Similar to the SO_{2n+1} case, I consider the Bessel functional on an unramified principal series of the group GSpin_{2n+1}. In the current stage, I grasped the techniques in [BFF97] and I see much of the work are reducible to the results over the orthgonal group. If anything is not directly reducible, it is very optimistic the techiques in *loc. cit.* will help.

4.2. **L-parameter of a simple supercuspidal representation: odd unitary group.** Local gamma factors of the odd unitary group has been studied in [CJ23], in particular, the coincidence of the Rankin-Selberg gamma factor and Shahidi's gamma factor is proven. The computation of the Rankin-Selberg gamma factor of a simple supercuspidal representation is promising.

4.3. **A follow up of [BX24].** The success of [AHKO23] on describing the L-parameters in the orthogonal group case over a dyadic field is a huge inspiration. It is very promising to have the endoscopic lift of a simple supercuspidal representation of the even unitary group, using the ideas and techniques in [AHKO23].

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